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THE RATE OF CONVERGENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES TO A STABLE LAW

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Abstract. In this paper we present uniform and nonuniform rates of convergence of sums of independent random variables to a stable law. The results obtained extend to the case of nonidentically distributed random variables considered by Hall [1].

1. Introduction. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables whose distribution functions $\{F_n, n \geqslant 1\}$ belong to the domain of attraction of a stable law of exponent α, $0 < \alpha < 2$. Then, under some additional assumptions, there exist constants $\{\tau_n, n \geqslant 1\}$ and $\{s_n, n \geqslant 1\}$ such that $\sum_{j=1}^{n} (X_j - \tau_j)/s_n$ converges to a stable law, as $n \to \infty$.

In this paper we present uniform and nonuniform convergence rates in this limit theorem. The former have been treated by many authors, mainly in the case of independent and identically distributed (i.i.d.) random variables (see, e.g., the work of Hall [1] and the reference therein). However, the nonuniform rates have not been so extensively studied even in the i.i.d. case. The presented paper, by our opinion, fills in this gap.

At first we extend Hall's [1] upper bounds to the nonidentically distributed random variables. The extension obtained is described in Theorem 1. Secondly, we derive the nonuniform rate of convergence of sums of independent nonidentically distributed random variables to a stable law with $1 < \alpha < 2$. These results are given in Theorem 2. As far as the authors know, Theorem 2 even in the i.i.d. case gives a new result.

We close this section with some notation. Let $\{\phi_n, n \geqslant 1\}$ be a sequence of the characteristic functions of $\{X_n, n \geqslant 1\}$ and let $\{c_{i,j}, j \geqslant 1\}$, $i = 1, 2$, be sequences of nonnegative numbers such that $c_{1,j} + c_{2,j} > 0$, $j \geqslant 1$. Write

$$h_j(x) = 1 - F_j(x) + F_j(-x) - (c_{1,j} + c_{2,j})(x^{-\alpha \wedge 1}),$$

$$d_j(x) = 1 - F_j(x) - F_j(-x) - (c_{1,j} - c_{2,j})(x^{-\alpha \wedge 1}),$$

$$H_n(x) = \sum_{j=1}^{n} h_j(x), \quad D_n(x) = \sum_{j=1}^{n} d_j(x),$$

$$\bar{H}_n(x) = \sum_{j=1}^{n} |h_j(x)|, \quad \bar{D}_n(x) = \sum_{j=1}^{n} |d_j(x)|.$$

Here and in what follows, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Let us put

$$e_1 = \int_0^\infty u^{-\alpha} \sin(u) du, \quad e_2 = \begin{cases} -\int_0^\infty u^{-\alpha}(\cos u) du & \text{if } \alpha \in (0, 1), \\ 1 & \text{if } \alpha = 1, \\ \int_0^\infty u^{-\alpha}(1-\cos u) du & \text{if } \alpha \in (1, 2), \end{cases}$$

$$b_{1,j}(t) = \int_0^\infty (1-\cos(tx)) dh_j(x) \quad \text{or} \quad b_{1,j}(t) = t \int_0^\infty \sin(tx) h_j(x) dx,$$

$$b_{2,j}(t) = \begin{cases} \int_0^\infty \sin(tx) dd_j(x) & \text{for } \alpha \in (0, 1), \\ t \int_0^\infty (1-\cos(tx)) d_j(x) dx & \text{for } \alpha \in [1, 2), \end{cases}$$

$$a_{1,j} = (c_{1,j} + c_{2,j}) e_1, \quad a_{2,j} = (c_{1,j} - c_{2,j}) e_2,$$

(1) $$\mu_j = \int_0^1 (1 - F_j(x) - F_j(-x)) dx + \int_1^\infty d_j(x) dx - (c_{1,j} - c_{2,j}) \gamma,$$

$$\tau_j = \begin{cases} \mu_j + \sum_{i=1}^{j-1} a_{2,i} \log(s_j/s_{j-1}) + a_{2,j} \log s_j & \text{if } \alpha = 1, \\ 0 & \text{otherwise } j \geqslant 1, \end{cases}$$

$$\gamma = \lim_{n\to\infty} \Big(\sum_{j=1}^{n} j^{-1} - \ln n\Big) \quad \text{(Euler's constant)},$$

$$C_{1,n} = \sum_{j=1}^{n} c_{1,j}, \quad C_{2,n} = \sum_{j=1}^{n} c_{2,j},$$

$$s_n^\alpha = (C_{1,n} + C_{2,n}) e_1, \quad s_0 = 0, \quad n \geqslant 1.$$

Throughout the paper, we assume that for some real number λ

(2) $$\lambda = \lim_{n\to\infty} (C_{1,n} - C_{2,n}) e_2 / (C_{1,n} + C_{2,n}) e_1, \quad \lim_{n\to\infty} s_n = +\infty$$

and, for $j \geqslant 1$, $EX_j = 0$ iff it exists.

Let $G_{\alpha,\lambda}(\cdot)$ denote the stable law with the characteristic function

$$\psi(t) = \begin{cases} \exp\{-|t|^{\alpha}(1+i\lambda \operatorname{sign} t)\} & \text{if } \alpha \neq 1, \\ \exp\{-|t|(1+i\lambda(\operatorname{sign} t)\ln|t|)\} & \text{if } \alpha = 1. \end{cases} \tag{3}$$

Let us define

$$\Delta_n(x) = |P[S_n < xs_n] - G_{\alpha,\lambda}(x)|, \quad \text{where } S_n = \sum_{j=1}^{n} (X_j - \tau_j).$$

In what follows, C denotes a positive constant which may only be dependent on α and λ.

2. The rates of convergence to a stable law. The following theorem presents the uniform rate of convergence of sums of independent nonidentically distributed random variables to a stable law.

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables, and let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of positive numbers such that, for every $0 \leqslant t \leqslant \eta$ and $n \geqslant 1$,*

$$\begin{aligned} &24 \max_{1 \leqslant j} \{t^{\alpha}(|a_{1,j}| + |a_{2,j}|) + |b_{1,j}(t)| + |b_{2,j}(t)|\} \leqslant 1 \quad \text{if } \alpha \neq 1, \\ &24 \max_{1 \leqslant j} \{t(|a_{1,j}| + |a_{2,j}(\log t)^2| + |\mu_j|) + |b_{1,j}(t)| + |b_{2,j}(t)|\} \leqslant 1 \quad \text{if } \alpha = 1, \end{aligned} \tag{4}$$

$$\Big|\sum_{j=1}^{n} a_{2,j}/s_n^{\alpha} - \lambda\Big| = \varepsilon_n, \tag{5}$$

where η is some positive number. Assume

$$\vartheta(x) = \sup_n \{\bar{H}_n(x) \vee \bar{D}_n(x)\} s_n^{-\alpha} = o(x^{-\alpha}) \text{ as } x \to \infty \quad \text{and} \quad \sup_x x^{\alpha}\vartheta(x) < \infty, \tag{6}$$

and, in addition, if $\alpha = 1$, then

$$\max_{1 \leqslant j} |b_{j,2}(t)|\,|\ln t| < \infty \quad \text{for } 0 < t < \eta. \tag{7}$$

(a) *If $\{h_n, n \geqslant 1\}$ and $\{d_n, n \geqslant 1\}$ are sequences of uniformly ultimately monotone functions on $[0, \infty)$ and $0 < \alpha < 1$, then*

$$\begin{aligned} \sup_x \Delta_n(x) \leqslant C\Big\{&s_n^{-2}\int_0^{s_n} x|H_n(x)|\,dx + s_n^{-1}\int_0^{s_n} |D_n(x)|\,dx \\ &+ \int_{s_n}^{\infty} x^{-1}\{|H_n(x)| + |D_n(x)|\}\,dx + \varepsilon_n + s_n^{-(\alpha \wedge (1-\alpha))}\Big\}. \end{aligned} \tag{8}$$

(b) *If* $\int_0^\infty \vartheta(x)dx < \infty$, *then for* $\alpha = 1$

(9) $$\sup_x \Delta_n(x) \leqslant C\{s_n^{-2} \int_0^{s_n} x|H_n(x)|dx + s_n^{-3} \int_0^{s_n} x^2|D_n(x)|dx + s_n^{-1} \int_{s_n}^{\infty} [|H_n(x)|+|D_n(x)|]dx + \varepsilon_n + s_n^{-1}(\log s_n)^2\}.$$

(c) *If* (b) *holds and* $\{h_n, n \geqslant 1\}$ *is a sequence of uniformly ultimately monotone functions on* $[0, \infty)$, *then for* $\alpha = 1$

(10) $$\sup_x \Delta_n(x) \leqslant C\{s_n^{-2} \int_0^{s_n} x|H_n(x)|dx + s_n^{-3} \int_0^{s_n} x^2|D_n(x)|dx + \int_{s_n}^{\infty} x^{-1}|H_n(x)|dx + s_n^{-1} \int_{s_n}^{\infty} |D_n(x)|dx + \varepsilon_n + s_n^{-2}(\log s_n)^2\}.$$

(d) *If* $\{h_n, n \geqslant 1\}$ *is a sequence of uniformly ultimately monotone functions, then for* $1 < \alpha < 2$

(11) $$\sup_x \Delta_n(x) \leqslant C\{s_n^{-2} \int_0^{s_n} x|H_n(x)|dx + s_n^{-3} \int_0^{s_n} x^2|D_n(x)|dx + \int_{s_n}^{\infty} x^{-1}|H_n(x)|dx + s_n^{-1} \int_{s_n}^{\infty} |D_n(x)|dx + \varepsilon_n + s_n^{\alpha-2}\}.$$

(e) *For* $1 < \alpha < 2$

(12) $$\sup_x \Delta_n(x) \leqslant C\{s_n^{-2} \int_0^{s_n} x|H_n(x)|dx + s_n^{-3} \int_0^{s_n} x^2|D_n(x)|dx + s_n^{-1} \int_{s_n}^{\infty} [|H_n(x)|+|D_n(x)|]dx + \varepsilon_n + s_n^{\alpha-2}\}.$$

Let us observe that the condition (4) plays the same role as Feller's condition in the central limit theorem. On the other hand, let us observe that if $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. random variables, then we easily get the upper bounds of Hall [1].

The nonuniform bounds are the following:

THEOREM 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent random variables satisfying the assumptions* (4)–(7) *of Theorem* 1. *If for every* $n \geqslant 1$ *the functions* $\bar{H}_n(x)$ *and* $\bar{D}_n(x)$ *are ultimately monotone on* $[0, \infty)$ *and* $1 < \alpha < 2$, *then for every* $x \in \boldsymbol{R}$

(13) $$(1+|x|)\Delta_n(x) \leqslant C\{s_n^{-1} \int_{s_n}^{\infty} (1+|\ln(x/s_n)|)[\bar{D}_n(x)+\bar{H}_n(x)]dx + \bar{H}_n(s_n) + s_n^{-3} \int_0^{s_n} x^2 \bar{D}_n(x)dx + s_n^{-2} \int_0^{s_n} x\bar{H}_n(x)dx + \varepsilon_n + s_n^{\alpha-2}\}.$$

3. Auxiliary lemmas and proofs. In the proofs of Theorems 1 and 2 we need five lemmas. Lemmas 1–3 are extensions (to nonidentically distributed random variables) of the corresponding ones given in [1]. Thus we omit the proofs.

LEMMA 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables. Then for every $j \geqslant 1$*

$$1-\phi_j(t) = \begin{cases} |t|^\alpha(a_{1,j}+ia_{2,j}\operatorname{sign} t)+b_{1,j}(t)+ib_{2,j}(t) \\ \quad +(c_{1,j}+c_{2,j})r_{1,\alpha}(t)+i(c_{1,j}-c_{2,j})r_{2,\alpha}(t) & \text{if } \alpha \neq 1, \\ |t|\big(a_{1,j}+i(\operatorname{sign} t)(-\mu_j+a_{2,j}\ln|t|)\big)+b_{1,j}(t)+ib_{2,j}(t) \\ \quad +(c_{1,j}+c_{2,j})r_{1,\alpha}(t)+i(c_{1,j}-c_{2,j})r_{2,\alpha}(t) & \text{if } \alpha = 1, \end{cases}$$

and for $1<\alpha<2$

$$\left|\frac{d}{dt}(\phi_j(t)-\psi_j(t))\right| \leqslant \left|\frac{d}{dt}b_{1,j}(t)\right|+\left|\frac{d}{dt}b_{2,j}(t)\right|+(c_{1,j}+c_{2,j})\left|\frac{d}{dt}r_{1,\alpha}(t)\right|$$
$$+(c_{1,j}-c_{2,j})\left|\frac{d}{dt}r_{2,\alpha}(t)\right|+\alpha|t|^{2\alpha-1}|a_{1,j}+ia_{2,j}|^2,$$

where

$$\psi_j(t) = \begin{cases} \exp\{-|t|^\alpha(a_{1,j}+ia_{2,j}\operatorname{sign} t)\} & \text{if } \alpha \neq 1, \\ \exp\{-|t|\big(a_{1,j}+ia_{2,j}(\operatorname{sign} t)\ln|t|\big)\} & \text{if } \alpha = 1, \end{cases}$$

$$r_{1,\alpha}(t) = 1-(\cos t)-t^\alpha\int_0^t \frac{\sin u}{u^\alpha}\,du = O(t^2) \quad \text{as } t\to 0,$$

$$r_{2,\alpha}(t) = \begin{cases} -(\sin t)+t^\alpha\displaystyle\int_0^t \frac{\cos u}{u^\alpha}\,du = O(t) & \text{if } 0<\alpha<1, \\ (t-\sin t)-t\big(\operatorname{ci}(t)-\gamma+\ln|t|\big) = O(t^3) & \text{if } \alpha = 1, \\ (t-\sin t)+t^\alpha\displaystyle\int_0^t \frac{1-\cos u}{u^\alpha}\,du = O(t^3) & \text{if } 1<\alpha<2, \end{cases}$$

as $t\to 0$, and

$$\operatorname{ci}(x) = \int_x^\infty \frac{\cos u}{u}\,du.$$

LEMMA 2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables and let*

$$B_{1,n}(t) = \sum_{j=1}^n b_{1,j}(t), \qquad B_{2,n}(t) = \sum_{j=1}^n b_{2,j}(t) \qquad n \geqslant 1.$$

If $0 \leqslant t \leqslant s_n(\frac{1}{8} \wedge \eta)$, *then for* $\alpha \neq 1$

$$|\prod_{j=1}^{n} \phi_j(t/s_n) - \prod_{j=1}^{n} \psi_j(t/s_n)| \leqslant |\prod_{j=1}^{n} \psi_j(t/s_n)| \{|B_{1,n}(t/s_n)| + |B_{2,n}(t/s_n)| + |R_n(t/s_n)|\} \exp\{|B_{1,n}(t/s_n)| + |R_n(t/s_n)|\} = I(n, t),$$

and for $\alpha = 1$

$$|\prod_{j=1}^{n} \phi_j(t/s_n) \exp\{-it(\mu_j + a_{2,j} \ln s_n)/s_n\} - \prod_{j=1}^{n} \psi_j(t/s_n) \exp\{-ita_{2,j}(\ln s_n)/s_n\}| \leqslant I(n, t),$$

where

$$|R_n(t/s_n)| \leqslant \tfrac{11}{12} \sum_{j=1}^{n} |1 - \phi_j(t/s_n)|^2 + s_n^\alpha \big(r_{1,\alpha}(t/s_n)/e_1 + r_{2,\alpha}(t/s_n)/e_2\big).$$

If $1 < \alpha < 2$ *and* $0 \leqslant t \leqslant s_n(\frac{1}{8} \wedge \eta)$, *then*

$$\left|\frac{d}{dt}\Big(\prod_{j=1}^{n} \phi_j(t/s_n) - \prod_{j=1}^{n} \psi_j(t/s_n)\Big)\right| \leqslant \Big\{(\alpha|t|^{\alpha-1}(1+\lambda)+1)\big(\bar{B}_{1,n}(t/s_n) + \bar{B}_{2,n}(t/s_n) + |t|^{2\alpha}/s_n^\alpha + s_n^\alpha(|r_{1,\alpha}(t/s_n)| + |r_{2,\alpha}(t/s_n)|)\big) + s_n^{-1} \sum_{j=1}^{n} \left|\frac{d}{du}\{-b_{1,j}(u) - ib_{2,j}(u) - (c_{1,j}+c_{2,j}) r_{1,\alpha}(u) - i(c_{1,j} - c_{2,j}) r_{2,\alpha}(u)\}\big|_{u=t/s_n}\right|\Big\} \exp\{-|t|^\alpha/4\},$$

where $\bar{B}_{j,n}(u) = \sum_{i=1}^{n} |b_{j,i}(u)|$, $j = 1, 2$, $n \geqslant 1$, $u \in \mathbf{R}$.

Proof. By Taylor's formulae for the function "log" and under the assumptions of Theorem 1, we have

$$|\prod_{j=1}^{n} \phi_j(t/s_n) - \prod_{j=1}^{n} \psi_j(t/s_n)| = |\prod_{j=1}^{n} \psi_j(t/s_n)|\, |1 - \exp\{\sum_{j=1}^{n} [-\sum_{k=1}^{\infty} (1-\phi_j(t/s_n))^k/k + \phi_j(t/s_n) - 1 - \ln \psi_j(t/s_n)]\}|.$$

Moreover, by (4),

$$|\sum_{k=1}^{\infty} (1-\phi_j(t/s_n))^k/k| \leqslant 2|1-\phi_j(t/s_n)|^2/3,$$

so that from Lemma 1 we obtain

$$\Big|\prod_{j=1}^{n}\phi_j(t/s_n)-\prod_{j=1}^{n}\psi_j(t/s_n)\Big|$$
$$=\Big|\prod_{j=1}^{n}\psi_j(t/s_n)\Big|\,\big|1-\exp\{R_n(t/s_n)-B_{1,n}(t/s_n)-iB_{2,n}(t/s_n)\}\big|.$$

Thus the inequalities $|e^{ix}-1|\leqslant|x|$, $|e^x-1|\leqslant|x|e^{|x|}$, $x\in\boldsymbol{R}$, complete the proof of the first part Lemma 2 in the case $\alpha\neq 1$. The proof for $\alpha=1$ runs similarly. By the inequalities

$$\Big|\frac{d}{dt}\Big(\prod_{j=1}^{n}\phi_j(t/s_n)-\prod_{j=1}^{n}\psi_j(t/s_n)\Big)\Big|\leqslant\sum_{j=1}^{n}\Big|\frac{d}{dt}\phi_j(t/s_n)\Big|\max_{1\leqslant k\leqslant n}\Big|\prod_{\substack{i=1\\ i\neq k}}^{n}\phi_i(t/s_n)-\prod_{\substack{i=1\\ i\neq k}}^{n}\psi_i(t/s_n)\Big|$$
$$+\sum_{j=1}^{n}\Big|\frac{d}{dt}(\phi_j(t/s_n)-\psi_j(t/s_n))\Big|\max_{1\leqslant k\leqslant n}\Big|\prod_{\substack{i=1\\ i\neq k}}^{n}\psi_i(t/s_n)\Big|,$$

$$\max_{1\leqslant k\leqslant n}\Big|\prod_{\substack{i=1\\ i\neq k}}^{n}\psi_i(t/s_n)\Big|\leqslant e^{-3|t|^\alpha/4},$$

$$\Big|\frac{d}{dt}(\phi_j(t/s_n)-\psi_j(t/s_n))\Big|\leqslant\alpha|t|^{2\alpha-1}|a_{1,j}+ia_{2,j}|^2 s_n^{-2\alpha}$$
$$+\Big|\frac{d}{du}\{-b_{1,j}(u)-ib_{2,j}(u)-(c_{1,j}+c_{2,j})r_{1,\alpha}(u)-i(c_{1,j}-c_{2,j})r_{2,\alpha}(u)\}\big|_{u=t/s_n}\Big|s_n^{-1},$$

$$\Big|\frac{d}{dt}\phi_j(t/s_n)\Big|\leqslant\alpha|t|^{2\alpha-1}|a_{1,j}+ia_{2,j}|s_n^{-\alpha}$$
$$+\Big|\frac{d}{du}\{-b_{1,j}(u)-ib_{2,j}(u)-(c_{1,j}+c_{2,j})r_{1,\alpha}(u)-i(c_{1,j}-c_{2,j})r_{2,\alpha}(u)\}\big|_{u=t/s_n}\Big|s_n^{-1}$$

and, as in the proof of the first part,

$$\max_{1\leqslant k\leqslant n}\Big|\prod_{\substack{j=1\\ j\neq k}}^{n}\phi_j(t/s_n)-\prod_{\substack{j=1\\ j\neq k}}^{n}\psi_j(t/s_n)\Big|$$
$$\leqslant\{\bar{B}_{1,n}(t/s_n)+\bar{B}_{2,n}(t/s_n)+|R_n(t/s_n)|\}\exp\{-3|t|^\alpha/4+\bar{B}_{1,n}(t/s_n)+|R_n(t/s_n)|\},$$

we obtain the second part of Lemma 2.

LEMMA 3. *Let the assumptions* (4)–(7) *of Theorem* 1 *be satisfied.*

(i) *If* $0<\alpha<1$ *and* $D_n(\cdot)$ *and* $H_n(\cdot)$ *are ultimately monotone on* $[0,\infty)$, *then for any* $c>0$

$$\int_0^\infty t^{-1}|B_{1,n}(t/s_n)|e^{-ct^\alpha}dt\leqslant C\Big\{s_n^{-2}\int_0^{s_n}x|H_n(x)|dx+\int_{s_n}^{\infty}x^{-1}|H_n(x)|dx\Big\},\tag{14}$$

and

$$\int_0^\infty t^{-1}|B_{2,n}(t/s_n)|e^{-ct^\alpha}dt \leqslant C\{s_n^{-1}\int_0^{s_n}|D_n(x)|\,dx+|D_n(s_n)|+\int_{s_n}^\infty x^{-1}|D_n(x)|\,dx\},$$

where

$$B_{1,n}(t) = -\int_0^\infty (1-\cos(tx))dH_n(x). \tag{15}$$

(ii) *If* $1 \leqslant \alpha < 2$ *and* $H_n(\cdot)$ *is ultimately monotone on* $[0, \infty)$ *and* $B_{1,n}(\cdot)$ *is defined by* (15), *then* (14) *holds. If* $1 \leqslant \alpha < 2$ *and* $B_{1,n}(\cdot)$ *is defined by the second formula in* (1), *then for every* $c > 0$ *and any* $\varepsilon > 0$

$$\int_0^\infty t^{-1}|B_{1,n}(t/s_n)|e^{-ct^\alpha}dt \leqslant C\{s_n^{-2}\int_0^{\varepsilon s_n} x|H_n(x)|\,dx+s_n^{-1}\int_{\varepsilon s_n}^\infty |H_n(x)|\,dx\},$$

and

$$\int_0^\infty t^{-1}|B_{2,n}(t/s_n)|e^{-ct^\alpha}dt \leqslant C\{s_n^{-3}\int_0^{\varepsilon s_n} x^2|D_n(x)|\,dx+2s_n^{-1}\int_{\varepsilon s_n}^\infty |D_n(x)|\,dx\}.$$

(iii) *If* $1 < \alpha < 2$ *and* $\bar{H}_n(\cdot)$ *and* $\bar{D}_n(\cdot)$ *are ultimately monotone functions, then for every* $c > 0$

$$\int_0^{\eta s_n} (t^{-2}+\alpha(1+\lambda)t^{\alpha-2})|\bar{B}_{1,n}(t/s_n)|e^{-ct^\alpha}dt$$
$$\leqslant C\{s_n^{-2}\int_0^{s_n} x|\bar{H}_n(x)|\,dx+s_n^{-1}\int_{s_n}^\infty |\bar{H}_n(x)|\,dx+s_n^{-1}\int_{s_n}^\infty |\ln(x/s_n)|\,|\bar{H}_n(x)|\,dx\},$$

$$\int_0^{\eta s_n} (t^{-2}+\alpha(1+\lambda)t^{\alpha-2})|\bar{B}_{2,n}(t/s_n)|e^{-ct^\alpha}dt$$
$$\leqslant C\{s_n^{-3}\int_0^{s_n} x^2|\bar{D}_n(x)|\,dx+2s_n^{-1}\int_{s_n}^\infty |\bar{D}_n(x)|\,dx\},$$

$$\int_0^{\eta s_n} t^{-1}(1+t^\alpha)s_n^{-1}\sum_{j=1}^n \left|\frac{d}{du}b_{1,n}(u)\right|\Bigg|_{u=t/s_n} e^{-ct^\alpha}dt$$
$$\leqslant C\{s_n^{-2}\int_0^{s_n} x|\bar{H}_n(x)|\,dx+\int_{s_n}^\infty x^{-1}|\bar{H}_n(x)|\,dx+s_n^{-1}\int_{s_n}^\infty |\ln(x/s_n)|\,|\bar{H}_n(x)|\,dx+\bar{H}_n(s_n)\},$$

$$\int_0^{\eta s_n} t^{-1}(1+t^\alpha)s_n^{-1}\sum_{j=1}^n \left|\frac{d}{du}b_{2,n}(u)\right|\Bigg|_{u=t/s_n} e^{-ct^\alpha}dt$$
$$\leqslant C\{s_n^{-3}\int_0^{s_n} x^2|\bar{D}_n(x)|\,dx+s_n^{-1}\int_{s_n}^\infty |\bar{D}_n(x)|\,dx+s_n^{-1}\int_{s_n}^\infty |\ln(x/s_n)|\,|\bar{D}_n(x)|\,dx\}.$$

LEMMA 4. *For every* $p > 0$, $0 < \alpha \leqslant 2$ *and* $\lambda_1 \in [-1, 1]$, $\lambda_2 \in [-1, 1]$,

$$\sup_x |G_{\alpha,\lambda_1}(x) - G_{\alpha,\lambda_2}(x)| \leqslant \big(\Gamma(1+1/\alpha)/(\pi\alpha)\big)|\lambda_1 - \lambda_2|, \tag{16}$$

$$(1+|x|)|G_{\alpha,\lambda_1}(x) - G_{\alpha,\lambda_2}(x)| \leqslant 640\big[2\Gamma\big((2\alpha-1)/\alpha\big) + (2+1/\alpha)\Gamma(1-1/\alpha)\big]|\lambda_1 - \lambda_2|/\pi \quad \textit{if } 1 < \alpha < 2, \tag{17}$$

$$\sup_x |G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \leqslant \big(p\Gamma(1/\alpha)/(\pi\alpha)\big) \wedge 1, \tag{18}$$

$$|G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \leqslant \Gamma(\alpha)\big||x+p|^{-\alpha} - |x|^{-\alpha}\big| \quad \textit{for } 0 < \alpha \leqslant 2,\ \alpha \neq 1, \tag{19}$$

$$\sup_x |G_{\alpha,\lambda_1}(px) - G_{\alpha,\lambda_1}(x)| \leqslant \begin{cases} (2/\pi\alpha)(p^\alpha \vee p^{-\alpha} + |\lambda_1|)|1 - p^\alpha \wedge p^{-\alpha}| & \textit{if } \alpha \neq 1, \\ (2/\pi)|1 - p^{-1} \wedge p|\big(p \wedge p^{-1} + |\lambda_1|\,|\gamma - 2\mathrm{ei}(-1)|\big) + (2/\pi)|\lambda_1|\,|\ln p| & \textit{if } \alpha = 1, \end{cases} \tag{20}$$

and

$$|G_{\alpha,\lambda_1}(px) - G_{\alpha,\lambda_1}(x)| \leqslant \Gamma(\alpha)|x|^{-\alpha}|p^{-\alpha} \vee p^\alpha - 1|\pi^{-1}, \tag{21}$$

where

$$\mathrm{ei}(x) = \int_{-\infty}^{x} (e^t/t)\,dt.$$

Proof. Let $g_{\alpha,\lambda}(x)$ denote the density function of the stable law with exponent α, $0 < \alpha \leqslant 2$ and $-1 \leqslant \lambda \leqslant 1$. Then (19) follows from the inequalities

$$|G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \leqslant \Big(\int_x^{x+p} g_{\alpha,\lambda_1}(u)\,du\Big) \wedge 1$$

and

$$g_{\alpha,\lambda_1}(u) \leqslant \Gamma(1/\alpha)/(\pi\alpha). \tag{22}$$

On the other hand, by Theorem 2.3.1 of [9, p. 100],

$$|G_{\alpha,\lambda_1}(x+p) - G_{\alpha,\lambda_1}(x)| \leqslant \int_x^{x+p} u^{-1-\alpha} g_{\alpha',\lambda'}(u^{-\alpha})\,du,$$

where $\alpha' = 1/\alpha$, $\lambda' = -1 + \alpha\big(1 + \lambda_1 K(\alpha)\big)$, $K(\alpha) = (\alpha - 1) + \operatorname{sign}(\alpha - 1)$. Then using (22) we get (19). Similarly one can get (21). Inequalities (17), (18) and (20) follow from Theorem 2 and Lemma 8 of [8, Chapter V].

LEMMA 5. *Let* h: $(0, \infty) \to [0, \infty)$ *be a function. If* $h(x)x^\alpha \to 0$ *as* $x \to \infty$ *for some* $\alpha > 0$, *then there exists a nonincreasing function* ε: $(0, \infty) \to (0, \infty)$ *such that* $\varepsilon(x) \to 0$, $x\varepsilon(x) \to \infty$ *and* $h\big(x\varepsilon(x)\big)x^\alpha \to 0$ *as* $x \to \infty$.

Proof. One can easily note that we may take

$$\varepsilon(x) = x^{-1/2} \vee \left(\sup\{h(y^{1/\alpha})y:\ y > x^{\alpha/2}\}\right)^{1/2\alpha}.$$

Proof of Theorem 1. In the proof we follow the ideas presented in [1] with necessary modifications needed for nonidentically distributed random variables.

At first, by the triangle inequality, we get

$$\sup_x \Delta_n(x) \leqslant \sup_x |P[\dot{S}_n < xs_n] - G_{\alpha,\lambda_n}(x)| + \sup_x |G_{\alpha,\lambda_n}(x) - G_{\alpha,\lambda}(x)|.$$

Thus, by Theorem 2 in [8, Chapter V], for every $T > 0$ and $\alpha \neq 1$ we have

$$\sup_x \Delta_n(x) \leqslant \pi^{-1} \int_0^T t^{-1} \Big|\prod_{j=1}^n \phi_j(t/s_n) - \prod_{j=1}^n \psi_j(t/s_n)\Big| dt + \sup_x |G_{\alpha,\lambda_n}(x) - G_{\alpha,\lambda}(x)| + \Gamma(1/\alpha)/(T\pi\alpha), \tag{23}$$

where

$$\lambda_n = \sum_{j=1}^n a_{2,j}/s_n^\alpha.$$

If $\alpha = 1$, then

$$\sup_x \Delta_n(x) \leqslant \pi^{-1} \int_0^T t^{-1} \Big|\prod_{j=1}^n \phi_j(t/s_n) \exp\{-it(\mu_j + a_{2,j} \ln s_n)/s_n\}$$
$$- \prod_{j=1}^n \psi_j(t/s_n) \exp\{-ita_{2,j}(\ln s_n)/s_n\}\Big| dt + \sup_x |G_{\alpha,\lambda_n}(x) - G_{\alpha,\lambda}(x)| + \Gamma(1/\alpha)/(T\pi\alpha).$$

Let us put $T = \eta s_n$. Let us remark that if we show that, for $\alpha \neq 1$,

$$\sup_n \left((|B_{1,n}(t)| + |B_{2,n}(t)|)s_n^{-\alpha}\right)t^{-\alpha} \to 0 \quad \text{as } t \to 0,$$

$$\sup_n (|R_n(t)|s_n^{-\alpha})t^{-\alpha} < 3/4 \quad \text{for sufficiently small } t,$$

and, for $\alpha = 1$,

$$\sup_n (|B_{1,n}(t)|s_n^{-1})t^{-1} \to 0 \quad \text{as } t \to 0,$$

$$\sup_n (|B_{2,n}(t)|s_n^{-1})t^{-1}|\ln t|^{-1} \to 0 \quad \text{as } t \to 0,$$

$$\sup_n (|R_n(t)|s_n^{-1})t^{-1} < 23/24 \quad \text{for sufficiently small } t,$$

then there exist constants $0 < c < 1$ and $0 < \theta < 1$ such that for $t \in (0, \eta\theta s_n)$ and every $\alpha \in (0, 2)$

$$\sup_n (|B_{1,n}(t/s_n)| + |R_n(t/s_n)|) < (1-c)t^\alpha. \tag{24}$$

Since we take supremum over all n, the constants c and θ do not depend on n.

Let $\varepsilon(x)$ be a function defined in Lemma 5 for the function $h(x) = \vartheta(x)$ given by (6). Of course, by (6) we have $x^\alpha \vartheta(x) \to 0$ as $x \to \infty$. For simplicity we put $\beta(x) = \varepsilon(1/x)$ and $\gamma(x) = \beta(x)/x$. We note that $\gamma(x) \to \infty$ as $x \to 0$.

Assume, e.g., that $0 < \alpha < 1$ is given. Then, by the Second Mean Value Theorem (h_j are uniformly ultimately monotone functions) and (6), we get

$$\begin{aligned}
\sup_n |B_{1,n}(t)| s_n^{-\alpha} t^{-\alpha} &\leqslant \sum_{j=1}^n |b_{1,j}(t)| s_n^{-\alpha} t^{-\alpha} \\
&\leqslant \sum_{j=1}^n \Big|\int_0^\infty (1-\cos tx)\,dh_j(x)\Big| s_n^{-\alpha} t^{-\alpha} \leqslant \sum_{j=1}^n t^{1-\alpha}\Big|\int_0^\infty (\sin tx) h_j(x)\,dx\Big| s_n^{-\alpha} \\
&\leqslant \sum_{j=1}^n t^{1-\alpha}\Big|\int_0^{\gamma(t)} (\sin tx) h_j(x)\,dx\Big| s_n^{-\alpha} + \sum_{j=1}^n t^{-\alpha}\Big|\int_{\beta(t)}^\infty (\sin u) h_j(u/t)\,du\Big| s_n^{-\alpha} \\
&\leqslant t^{2-\alpha}\Big(\int_0^{\gamma(t)} x\bar{H}_n(x)\,dx\Big) s_n^{-\alpha} + t^{-\alpha}\bar{H}_n(\gamma(t)) s_n^{-\alpha} \\
&\leqslant t^{2-\alpha} \sup_x \sup_n \big(x^\alpha \bar{H}_n(x) s_n^{-\alpha}\big) \int_0^{\gamma(t)} x^{1-\alpha}\,dx + t^{-\alpha} \sup_n \big(\bar{H}_n(\varepsilon(1/t)/t) s_n^{-\alpha}\big) \\
&\leqslant C\{(t\gamma(t))^{2-\alpha} + \vartheta(\varepsilon(1/t)/t) t^{-\alpha}\} \\
&\leqslant C\{(\varepsilon(1/t))^{2-\alpha} + \vartheta(\varepsilon(1/t)/t) t^{-\alpha}\} \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

Furthermore, similarly we obtain

$$\begin{aligned}
&\sup_n |B_{2,n}(t)| s_n^{-\alpha} t^{-\alpha} \leqslant \sum_{j=1}^n |b_{2,j}(t)| s_n^{-\alpha} t^{-\alpha} \\
&\leqslant \sum_{j=1}^n \Big|\int_0^\infty (\cos tx) d_j(x)\,dx\Big| s_n^{-\alpha} t^{-\alpha} \\
&\leqslant \sum_{j=1}^n t^{1-\alpha}\Big|\int_0^{\gamma(t)} (\cos tx) d_j(x)\,dx\Big| s_n^{-\alpha} + \sum_{j=1}^n t^{-\alpha}\Big|\int_{\beta(t)}^\infty (\cos u) d_j(u/t)\,du\Big| s_n^{-\alpha} \\
&\leqslant t^{1-\alpha} \sup_x \sup_n \big(x^\alpha \bar{D}_n(x) s_n^{-\alpha}\big) \int_0^{\gamma(t)} x^{-\alpha}\,dx + t^{-\alpha} \sup_n \big(\bar{D}_n(\varepsilon(1/t)/t) s_n^{-\alpha}\big) \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

Let us remark that we have proved more that it was needed. We prove that

$$\sup_n \Big\{ \sum_{j=1}^n \big(|b_{1,j}(t)| + |b_{2,j}(t)|\big) s_n^{-\alpha} \Big\} t^{-\alpha} \to 0 \quad \text{as } t \to 0$$

while it is sufficient to show that

$$\sup_n \Big\{\Big| \sum_{j=1}^n b_{1,j}(t)\Big| s_n^{-\alpha}\Big\} t^{-\alpha} \to 0 \quad \text{as } t \to 0.$$

The stronger version will be needed in the estimations of

$$\sup_n \{|R_n(t)|s_n^{-\alpha}\} t^{-\alpha} \quad \text{for } \alpha = 1 \text{ and } \alpha \neq 1.$$

By Lemmas 1, 2 and (4)–(7), we have

$$\begin{aligned}|R_n(t)s_n^{-\alpha}t^{-\alpha}| &\leqslant \tfrac{11}{6}\max_{1\leqslant j}|1-\phi_j(t)-(c_{1,j}+c_{2,j})r_{1,\alpha}(t)t^\alpha \\ &\quad -i(c_{1,j}-c_{2,j})r_{2,\alpha}(t)t^\alpha|\{\sum_{j=1}^n (|b_{1,j}(t)|+|b_{2,j}(t)|)+s_n^\alpha t^\alpha\} s_n^{-\alpha}t^{-\alpha} \\ &\quad +\tfrac{11}{3}\max_{j\geqslant 1}(a_{1,j}+a_{2,j})(r_{1,\alpha}^2(t)+r_{2,\alpha}^2(t))/(t^\alpha(e_1^2\wedge e_2^2)) \\ &\quad +(|r_{1,\alpha}(t)|+|r_{2,\alpha}(t)|)/(t^\alpha(e_1\wedge e_2)) \\ &\leqslant \tfrac{11}{24}\sup_n\{\sum_{j=1}^n (|b_{1,j}(t)|+|b_{2,j}(t)|)s_n^{-\alpha}\}t^{-\alpha}+\tfrac{11}{24}+o(t)\leqslant \tfrac{3}{4}\end{aligned}$$

for sufficiently small t.

The other inequalities in the proof of (24) can be obtained similarly. By (24), for some $c > 0$ we get

$$|\prod_{j=1}^n \psi_j(t/s_n)| \exp\{|B_{1,n}(t/s_n)|+|R_n(t/s_n)|\} \leqslant \exp\{-c|t|^\alpha\}$$

and, by (23),

$$\begin{aligned}\sup_x \Delta_n(x) &\leqslant \pi^{-1}\int_0^{\theta\eta s_n} t^{-1}(|B_{1,n}(t/s_n)|+|B_{2,n}(t/s_n)|)e^{-ct^\alpha}dt+C\varepsilon_n+Cs_n^{-1} \\ &\quad +\pi^{-1}\int_0^{\theta\eta s_n} t^{-1}|R_n(t/s_n)|e^{-ct^\alpha}dt.\end{aligned}$$

By Lemma 3 we may estimate the first term on the right-hand side of the last inequality. The last term, by (4), Lemmas 2 and 1 can be bounded by

$$\begin{aligned}2^6\pi^{-1}&\int_0^{\theta\eta s_n} t^{-1}\{(\sum_{j=1}^n |t|^{2\alpha}/s_n^{2\alpha}|a_{1,j}+ia_{2,j}|^2+b_{1,j}^2(t/s_n)+b_{2,j}^2(t/s_n) \\ &\quad +s_n^\alpha \max_{1\leqslant j\leqslant n}(c_{1,j}+c_{2,j})(r_{1,\alpha}^2(t/s_n)+r_{2,\alpha}^2(t/s_n)))\}e^{-ct^\alpha}dt \\ &\leqslant C\int_0^{\theta\eta s_n} t^{2\alpha-1}s_n^{\alpha-1}(\eta^{-1}+1)e^{-ct^\alpha}dt+C\int_0^{\theta\eta s_n} t^{-1}s_n^\alpha((t/s_n)^2+(t/s_n)^{2[\alpha]+1} \\ &\quad +(t/s_n)^4+(t/s_n)^{4[\alpha]+1})e^{-ct^\alpha}dt\leqslant Cs_n^{-(\alpha\wedge(2[\alpha]+1-\alpha))},\end{aligned}$$

which completes the proof of Theorem 1 in the case $0 < \alpha < 1$. In the case $\alpha = 1$ or $1 < \alpha < 2$ the proof of Theorem 1 runs similarly, so we omit the details.

Proof of Theorem 2. By assumptions and Lemmas 7 and 8 in [8, Chapter VI], for every $T > 0$ we have

$$(25) \qquad (1+|x|)\Delta_n(x) \leqslant C\int_0^T t^{-2}\Big|\prod_{j=1}^n \phi_j(t/s_n) - \prod_{j=1}^n \psi_j(t/s_n)\Big|dt$$
$$+C\int_0^T t^{-1}\Big|\prod_{j=1}^n \phi_j(t/s_n) - \prod_{j=1}^n \psi_j(t/s_n)\Big|dt$$
$$+C\int_0^T t^{-1}\left|\frac{d}{du}\Big(\prod_{j=1}^n \phi_j(u) - \prod_{j=1}^n \psi_j(u)\Big)\Big|_{u=t/s_n}\right| s_n^{-1}\,dt$$
$$+|G_{\alpha,\lambda_n}(x) - G_{\alpha,\lambda}(x)| + C/T.$$

Since the assumptions of Theorem 2 imply the ones of Theorem 1 (e), we may estimate the second term of the right-hand side of (25) by the right-hand side of (12). Now Lemmas 1–4 and (25) give Theorem 2.

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